# CHAPTER 3:

## Decision Trees

# Section 3.0 Decision Trees

We all make decisions in our jobs, our communities and in our personal lives that involve significant uncertainty. Some examples are:

- How much technology should be invested in a plant when product demand is uncertain?
- How much car collision insurance should be purchased when you do not know whether or not you will have an accident?
- Should a company build a new power plant in a foreign country and, if yes, in which country?
- On a personal financial level, how much should you invest in a particular stock or mutual fund?
- Should millions be invested in a new drug that has proven effective in animal tests?
- How much time should you invest in studying when the questions on the exam are uncertain and payback from more study-time is hard to predict?
- What career should you pursue when the economy and job market are uncertain?

Specific stocks and the stock market as a whole demonstrate much uncertainty from day to day and year to year. There is uncertainty in the demand for power and the stability of developing countries. The link between animal drug trials and drug effectiveness in humans is far from perfect. Individuals experience different types of random accidents.

Decision analysis is an operations research modeling tool used to select the best decision in the presence of uncertainty.

- What specific uncertainties do you face in the next day, week, or month?
- What about other members of your family or friends?
- What uncertainties will affect your planning over the next year?

The oil industry was one of the earliest users of decision analysis and continues to lead in its application. Pharmaceutical companies apply decision trees, which are an extension of probability trees, to make research and development decisions. Industrial giants such as DuPont, Xerox, and Kodak have used decision trees to plan new products and production capacity. The US Forest Service uses decision trees to plan controlled forest fires. The Decision Analysis Affinity Group (www.daag.org) is an organization that runs conferences at which corporate users of decision analysis share experiences. There is also the Decision Analysis Society which is an organization affiliated with the Institute for Operations Research and the Management Sciences (INFORMS). They maintain a homepage at https://www.informs.org/Community/DAS

The decision tree methodology involves accounting for every possible decision and random outcome. The best alternative generally maximizes the ***expected value*** of profit or minimizes the expected value of cost. Sometimes the variable optimized is not financial. The goal might be to minimize time to complete a task or maximize the number of people attending an event. Modern software such as Precision Tree, an Excel add-on, helps with the analysis and offers visual representations of the results. These enable a decision maker to explore the strengths and weaknesses of the alternatives.

As decision analysis developed, the leaders in the field recognized two critical psychological and practical issues that needed to be addressed in order to make the tool of greater practical value. First, the models required estimates of probabilities that were often not easy to obtain with a detailed analysis of data. Therefore, subject matter experts were interviewed in order to estimate the probabilities. Decision analysts, along with mathematical psychologists, became leaders in the effort to understand biases and misconceptions that individuals display when asked to make a forecast. They developed interview protocols in order to obtain expert opinion in a way that reduces the likely bias.

Second, the expected value does not capture the fact that people are often fearful of taking risks, especially large ones. This risk aversion is the foundation for all of the insurance industry and the huge market in extended warranties. Decision analysts became leaders in researching attitudes towards risk and designing a methodology called utility theory that captures this behavior.

## Section 3.1 Select Prom Location

Lily Trump and Donald Tomlin were recently appointed as co-chairs for this year's senior prom. One of the big budget items is the rental of a hall. They visited four potential locations and obtained the cost and size data in Table 3.1.1. The Petty Hall was the smallest and cheapest. It would cost $1,500 and had dancing room capacity for 420 people. The largest was the Grand Hall which would cost $2,600 but could handle as many as 515 prom goers. Providence and Independence Halls were in the midsize and middle price range.

| Name | Normal Capacity | Cost |
|---|---|---|
| Petty Hall | 420 | $1,500 |
| Providence Hall | 455 | $1,800 |
| Independence Hall | 490 | $2,200 |
| Grand Hall | 515 | $2,600 |

**Table 3.1.1:** Hall capacity and cost

This year's senior class of 525 students is the largest ever. Lily and Donald worried about the possibility of every senior coming and some bringing outside guests. If that happened, even the largest hall visited would not be large enough. They also worried that if only 50% of the students attended they would be wasting a lot of money and the hall would look empty. They approached Ms. Karin Topper, the faculty advisor for the prom.

Ms. Topper had been advising prom committees for more than 15 years. She explained to the co-chairs that they should apply probability in thinking about their decision. They would have to balance the risk of not having enough room against the cost of paying for unneeded space.

Ms. Topper created an influence diagram to highlight the issue of key random events (see Figure 3.1.1).There were two main uncertainties. One uncertainty was the percent of seniors who plan to attend. The second uncertainty related to the proportion of seniors who will bring dates who are either not seniors or who attend another school.  These two numbers will affect whether or not a hall will be adequate and if they will have to turn away some ticket requests.

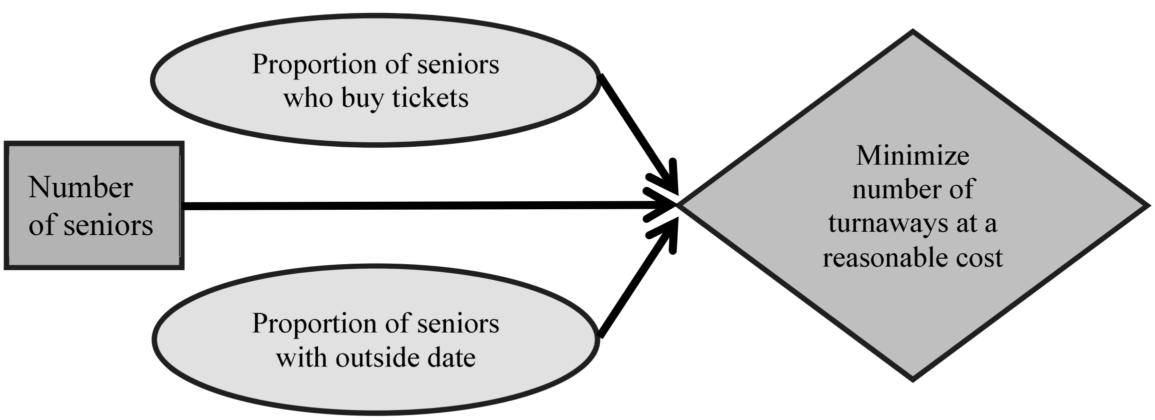



**Figure 3.1.1:** Influence diagram of the key random events

1. What are some reasons why a person would choose not to attend his or her senior prom?

2. At your school, what proportion of seniors would you expect not to attend prom?

3. What information could you use to estimate how many of the seniors who will attend the prom will also bring a date from outside the senior class?

Ms. Topper explained that based on her experience she could provide probability estimates for each of these two proportions. She believed that at least two-thirds of the students would attend although it could be slightly less. She thought that it was unlikely that more than four out of five would attend. She also added to the list of possibilities an intermediate rate of three-fourths of the seniors attending. In thinking about these three rates, Ms. Topper felt that with the down economy, the two-thirds and three-quarters rates were equally likely. The probability of an attendance rate as high as four out of five attending was half as likely as each of the other two rates.

Ms. Topper translated her reasoning into probabilities as follows. She let $X$ represent the probability that the proportion attending will be two-thirds. The same value $X$ would also apply to the three-quarters estimate. Lastly, the probability that the proportion would be four-fifths was one half of $X$ or $0.5X$. She explained that the sum of the probabilities of the various outcomes must be 1. She used the following equation to solve for $X$.

$$\begin{aligned} X + X + 0.5X &= 1 \\ 2.5X &= 1 \\ X &= 0.4 \end{aligned}$$

Based on her experience, she estimated there was a 0.4 probability of two-thirds of the seniors attending and a similar probability that three-fourths would attend. Ms. Topper estimated there was only a 0.2 probability that as many as four out five seniors would attend. These estimates are summarized in Table 3.1.2.

| Rate of seniors attending prom | Estimated probability |
|---|---|
| Two out of three | 0.4 |
| Three out of four | 0.4 |
| Four out of five | 0.2 |

**Table 3.1.2:** Estimates of various rates of seniors attending prom

In thinking about 15 years of experience, she recalled that as few as 10% of the seniors brought dates from outside the senior class and that this occurred less than half the time. However, at the other extreme, the rate of 20% occurred slightly more frequently. She assigned a 0.45 probability to the event that 10% of the attending seniors brought outside guests and a 0.55 probability to the event that 20% brought outside guests.

Lily and Donald were somewhat skeptical of Ms. Topper's estimates. They asked if she had any data to back up her expert opinion. Ms. Topper told the students to speak with Ms. Sophia Numerati, the school registrar. She was known to collect all kinds of data. Ms. Numerati had in fact collected data on the proms for the last 15 years. She provided the data that appear in Table 3.1.3. Lily and Donald looked at the table. The data just looked like a bunch of numbers on a page. They needed a way to make sense of the data if they were going to use them to help make the best possible decision about which hall to rent.

| Year | Class size | Seniors attendees | Percent of class | Outside guests | Percent of senior attendees with outside guests | Total attendance |
|---|---|---|---|---|---|---|
| 1997 | 443 | 328 | 74% | 59 | 18% | 387 |
| 1998 | 476 | 319 | 67% | 29 | 9% | 348 |
| 1999 | 499 | 339 | 68% | 64 | 19% | 404 |
| 2000 | 446 | 290 | 65% | 64 | 22% | 354 |
| 2001 | 435 | 348 | 80% | 31 | 9% | 379 |
| 2002 | 486 | 369 | 76% | 78 | 21% | 447 |
| 2003 | 474 | 384 | 81% | 77 | 20% | 461 |
| 2004 | 454 | 341 | 75% | 27 | 8% | 368 |
| 2005 | 420 | 307 | 73% | 55 | 18% | 362 |
| 2006 | 494 | 336 | 68% | 34 | 10% | 370 |
| 2007 | 475 | 361 | 76% | 40 | 11% | 401 |
| 2008 | 464 | 302 | 65% | 60 | 20% | 362 |
| 2009 | 493 | 325 | 66% | 33 | 10% | 358 |
| 2010 | 478 | 359 | 75% | 75 | 21% | 434 |
| 2011 | 457 | 361 | 79% | 43 | 12% | 404 |

**Table 3.1.3:** Senior prom data, 1997-2011

1. Which year had the highest number of seniors attending the prom?
2. What were the highest and lowest percentages of seniors attending the prom?
3. What was the largest fluctuation in the number of outside guests from one year to the next?
4. Within the past five years, what is the range of total attendance at the prom?
5. Does there appear to be a relationship between the percentage of seniors who attend prom and the percentage of seniors who bring an outside guest?
6. Based on the data in the table, what is your best guess about how many people will attend the prom this year?

Noting their confusion, Ms. Numerati suggested that they first focus on the column that recorded the percent of seniors who attended. Perhaps they would see some patterns.

| Senior Class | |
|---|---|
| **Percent of Class** | **Grouped Average** |
| 65% | 66.5% |
| 65% | |
| 66% | |
| 67% | |
| 68% | |
| 68% | |
| 73% | 74.8% |
| 74% | |
| 75% | |
| 75% | |
| 76% | |
| 76% | |
| 79% | 80% |
| 80% | |
| 81% | |

**Table 3.1.4:** Percentage of seniors at the Prom

After reviewing the sorted data, they noticed a cluster of values in the mid-60s and another one in the mid-70s. They grouped the six lowest values and found the average for that group of six classes. The average percentage was 66.5%, which was very close to Ms. Topper's estimate of two-thirds. They grouped the values in the mid-70s and those percentages averaged out to be 74.8%, or almost three-fourths. The three highest values were all close to 80%. They used the relative frequencies for each of the categories to estimate the proportion of times a rate close to two-thirds or three-fourths was reported. These proportions could be used to estimate the likelihood or probability of that approximate rate occurring.

> **Relative Frequency and Probability:**
> There is strong two-way link between the relative frequency and probability. Analysts routinely use the relative frequency of something happening to estimate its probability. Conversely, the probability of an event happening such as rolling a seven in dice, will equal the long-term relative frequency of observing a seven.

7. How did the percentages compare to Ms. Topper's expert judgment?

They sort ordered the percentages of guests (i.e., non seniors or students from other schools) who attended as in Table 3.1.5. There once again seemed to be clusters around the 10% and 20% values mentioned by Ms. Topper.

| **Guests** | |
|---|---|
| **Percent of senior attendees with guests** | **Grouped Average** |
| 8% | |
| 9% | |
| 9% | |
| 10% | |
| 10% | |
| 11% | |
| 12% | |
| 18% | |
| 18% | |
| 19% | |
| 20% | |
| 20% | |
| 21% | |
| 21% | |
| 22% | |

**Table 3.1.5:** Percentage of seniors who brought outside guests

8. What was the average value for each group of data?

9. What were the associated frequencies for each group? How closely did they agree with Ms. Topper's probability estimates?

Multiply probability of two separate events: It is allowed to multiply two probabilities when the two events being modeled are independent of one another. We are assuming in this case that the percentage of seniors who choose to attend the prom does not in any way affect the percentage of seniors who will invite outside guests.

Ms. Topper suggested that the co-chairs create a probability tree to visualize the six different combinations of outcomes. They are considering only three possibilities for the proportion of seniors attending and two possibilities for the percent of outside guests they bring. The top path in Figure 3.1.2 represents the random outcome that two-thirds of the seniors attend and

10% of them bring outside dates. The probability of that happening is 0.4 × 0.45 which equals 0.18. The total number of attendees in this instance is calculated below.

$$(525)\left(\frac{2}{3}\right)+(525)\left(\frac{2}{3}\right)(0.1)=385$$

Now consider the second path. This path involves the random sequence that two-thirds of the seniors attend and 20% of them bring guests.

10. Show that the probability of this occurring is 0.22 and the total number of attendees would be 420.

11. Calculate the probability and outcome of each of the remaining paths in Figure 3.1.2.

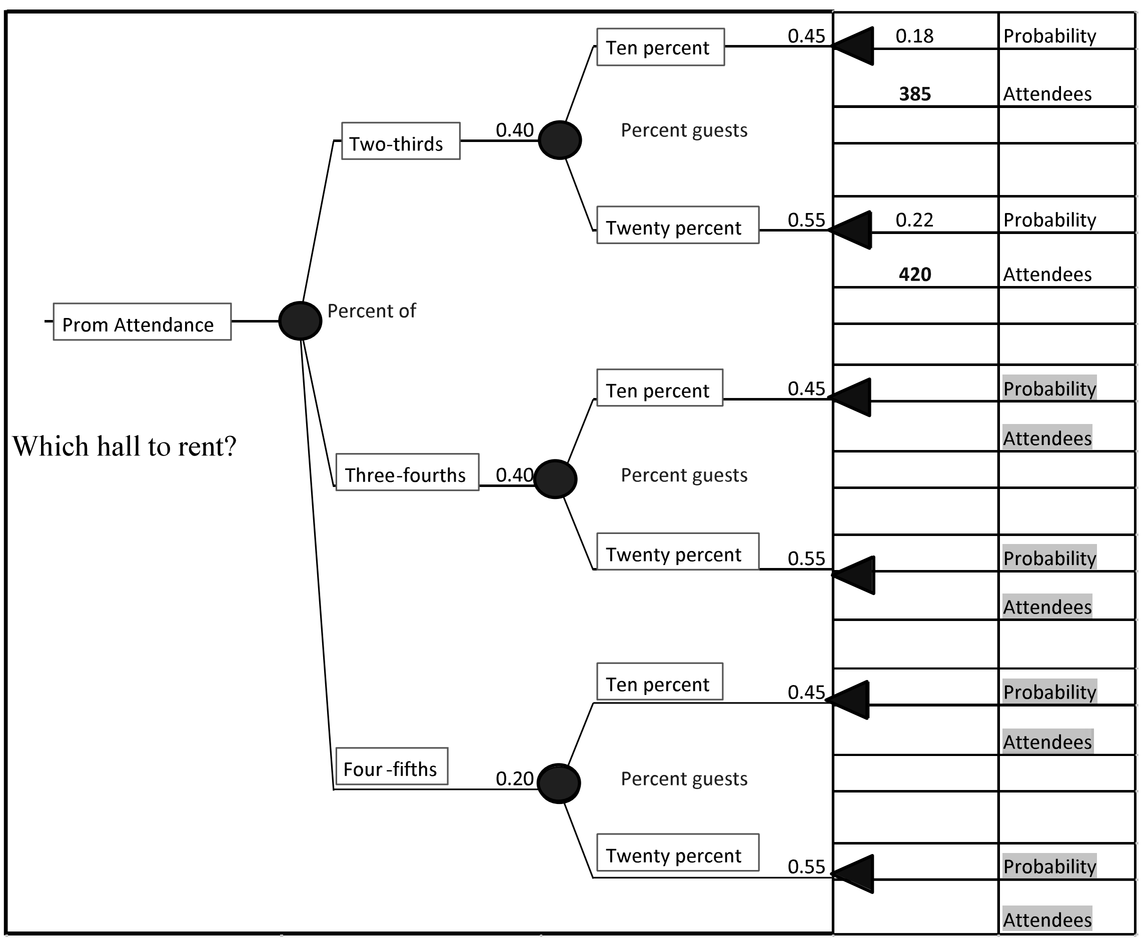



**Figure 3.1.2:** Probability tree of the two random events

Lily and Donald took all of the end values and their corresponding probabilities and organized them in Table 3.1.6. Before filling in the table, they sort-ordered the outcomes. The lowest value was 385 attendees and the highest was 504. In the next column they placed the associated probability. They then calculated the cumulative probabilities and recorded them in the final column. This column is interpreted as follows. There is a 0.40 probability that the number of attendees will be 420 or fewer, because 0.18 + 0.22 = 0.40. Similarly, there is a 0.89 probability that the total number of attendees will be 473 or fewer, because 0.18 + 0.22 + 0.18 + 0.09 + 0.22 = 0.89. If the estimates provided by Ms. Topper are accurate, it is certain that there will not be more than 504 attendees.

| **Number of Attendees** | **Probability** | |
|---|---|---|
| | **Relative** | **Cumulative** |
| 385 | 0.18 | 0.18 |
| 420 | 0.22 | 0.40 |
| 433 | 0.18 | 0.58 |
| 462 | 0.09 | 0.67 |
| 473 | 0.22 | 0.89 |
| 504 | 0.11 | 1 |

**Table 3.1.6:** Probability distribution of the number of guests

Ms. Topper suggested that Lily and Donald calculate the expected or average value of the ***random variable***, the number of attendees. Any variable represents a quantity that can take on different values. The value of a random variable is subject to variations due to chance.

12. Explain why the number of people attending the prom is a random variable.

The ***expected value*** of a random variable is calculated by multiplying each possible outcome by its probability and adding all these terms together. The formula for expected value of a random variable with six different outcomes is

$$\begin{aligned}E(X) &= \sum_{i=1}^{6} x_i \cdot P(X = x_i)\\ &= x_1 \cdot P(X = x_1) + x_2 \cdot P(X = x_2) + x_3 \cdot P(X = x_3) + x_4 \cdot P(X = x_4)\\ &\qquad + x_5 \cdot P(X = x_5) + x_6 \cdot P(X = x_6)\\ &= (385)(0.18) + (420)(0.22) + (433)(0.18) + (462)(0.09) + (473)(0.22) + (504)(0.11)\\ &\approx 441 \text{ attendees}\end{aligned}$$

The notation $E(X)$ is interpreted as "the expected value of $X$," where $X$ is a random variable. The terms in the formula are of the form $x_i \cdot P(X = x_i)$, where $P(X = x_i)$ is the probability of the random variable having the value $x_i$.

After seeing this expected value, they were ready to sign a contract with Providence Hall. Its capacity was 455 people which was higher than the expected value of 441. However, Ms. Topper cautioned them to take a closer look at the cumulative probabilities in the final column of Table 3.1.6. She asked them to consider the following questions.

13. If they rent Independence Hall, what is the probability that it will be able to handle all of those who want to attend with their guests?

14. If they rent Independence Hall, what is the probability that they will have to turn away classmates who want to buy tickets to the prom?

They decided to use the information in Table 3.1.4 and add a new column to Table 3.1.1. This column would compare the capacity to the cumulative probabilities. See Table 3.1.7.

| Name | Normal Capacity | Cost | Probability it will be adequate |
|---|---|---|---|
| Petty Hall | 420 | $1,500 | 0.4 |
| Providence Hall | 455 | $1,800 | 0.58 |
| Independence Hall | 490 | $2,200 | 0.89 |
| Grand Hall | 515 | $2,600 | 1 |

**Table 3.1.7:** Hall options and likelihood that it is adequate

15. Which hall should Donald and Lily rent? Justify your recommendation.

# Section 3.2 Boss Controls Invests in Automation

Boss Controls (BC) is an automotive supplier. It manufactures integrated cup holders with temperature controls for car-makers throughout the world. Their new model is to be made available on 1,000,000 new luxury cars worldwide. Initial estimates that car buyers will select this option are as high as 50% of the time or as low as 30% of the time. Because of a general decline in the global economy, BC marketing estimates there is a slightly higher chance that demand will be 30% rather than 50%. Management assigns a 0.55 probability that only 30% of the luxury car buyers will request their temperature controlled cup holders. The only other possibility they are considering is that 50% of luxury car buyers will request this option. They assign a 0.45 probability to that possibility.

1. Why did they have to assign a 0.45 probability to the second possibility?

## 3.2.1 Decision

Like most companies, BC wants to maximize its profit. BC's revenue comes from selling their cup holders to car companies at a fixed price of $60 apiece. BC can control manufacturing costs by how heavily it invests in automation. An investment of $13 million in high-speed equipment would lower the cost of producing each cup holder to $12. With a more modest investment of $8 million, the cost would be $27 per cup holder. Therefore, the net profit per unit after a high or low investment in automation will be $48 or $33, respectively. This per unit profit does not yet include the investment cost.

2. What decision must be made in this situation?

3. What is the ***chance event*** that BC faces? (i.e., an event whose likelihood must be predicted using probability and is outside the control of BC's management)?

## 3.2.2 Decision Tree to Solve the BC Problem

In situations such as this, managers need to take a systematic approach to determine the best strategy for achieving their goal. We will build a ***decision tree*** similar to the probability tree in the last section to model the situation. The tree will show all the possible outcomes, their probabilities and their consequences.

Decision trees are made of ***nodes*** and ***arcs***. The nodes on a decision tree represent decisions or random events. The two types of nodes, decision and random chance, are represented differently in a decision tree. Decision nodes are represented as rectangles as in Figure 3.2.1. Random chance nodes as circles as in Figure 3.2.2. There are arcs that connect these nodes. A sequence of nodes and arcs is a path that represents a possible scenario that can result from a specific decision and a random event.

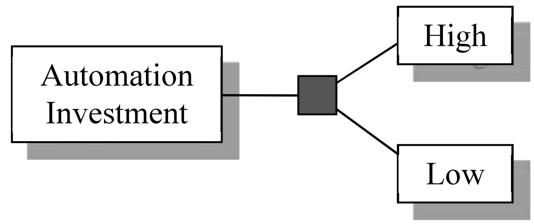



**Figure 3.2.1:** Decision node

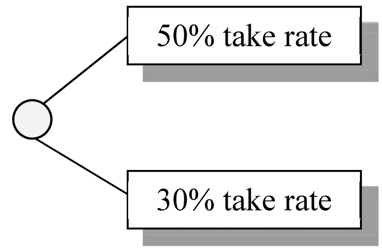



**Figure 3.2.2:** Random event node

In the BC scenario, whether there will be a high or low level of investment is a *decision* that management must make. However, whether the demand for their product will be 50% or 30% is strictly a *random event*; BC's management has no control over it. The two elements, decision and random chance, can be combined into a decision tree that models the entire Boss Control scenario, as shown in Figure 3.2.3.

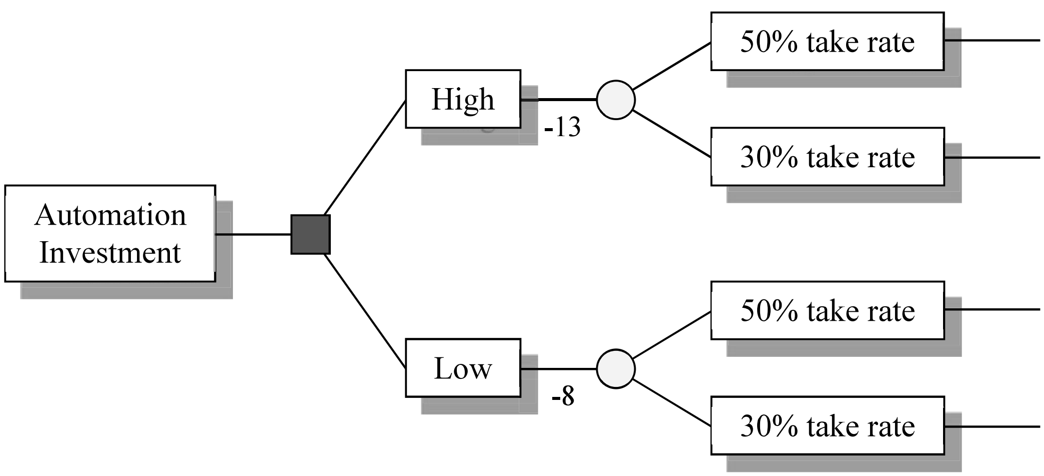



**Figure 3.2.3**: Decision tree for BC automation investment

This decision tree shows all the possible outcomes for every combination of decisions and random events. Now we must add the appropriate data to determine the expected profit of each path through the decision tree.

Recall that the low investment in automation was going to cost BC $8 million; the high investment was going to cost $13 million. This information can be added to the decision tree by placing the numbers along the appropriate branches, below the arcs. (The space above the arcs is reserved for another quantity, as you will see shortly). In this problem, the units on all such numbers will be millions of dollars. The numbers below are negative because they represent expenses BC must pay, not revenue earned.

Recall also that the unit profit after a low investment in automation is $33 whereas the unit profit after a high investment is $48. The "30%" and "50%" on the arcs after the chance nodes represent the proportion of car buyers who select the luxury cup holder option. This is referred to as the *take rate*. In BC's case, the estimated low take rate is 30% and the estimated high take rate is 50%.

4. What is the estimated probability that the take rate will be 30%?

5. What is the estimated probability that the take rate will be 50%?

6. What is the base of these take rates? In other words, what is the total number of potential sales?

The net revenue for each case can be calculated by multiplying the projected number of buyers times the net profit per unit. The projected net revenue with a 50% take rate and *high* investment is

Net Revenue = (1,000,000)×(0.5)×($48) = $24 million.

7. What is the projected net revenue with a 30% take rate and *high* investment?

8. What is the projected net revenue with a 50% take rate and *low* investment?

9. What is the projected net revenue with a 30% take rate and *low* investment?

In Figure 3.2.4, the earlier decision tree has been updated to include the information above. As before, the numbers beneath the arcs are in millions of dollars. They represent cost or net revenue. The numbers assigned above the arcs coming from a random event show the probability of that path.

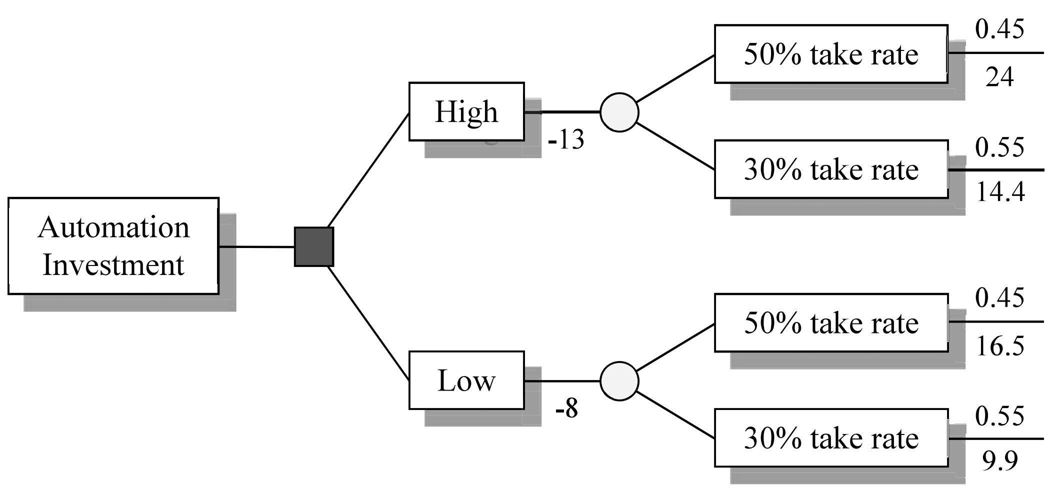



**Figure 3.2.4:** Boss Controls decision tree showing dollar values and probabilities

An end node, represented by a triangle, is placed at the end of each path through the decision tree. Next to the triangle, we record the net profit for that path. The net profit is determined by subtracting the investment cost from the net revenue. For example, with a high investment of $13 million and a 50% take rate, the net profit is

$24 million – $13 million = $11 million.

Figure 3.2.5 has been updated to show each of the four possible net profit values to the right of the end nodes.

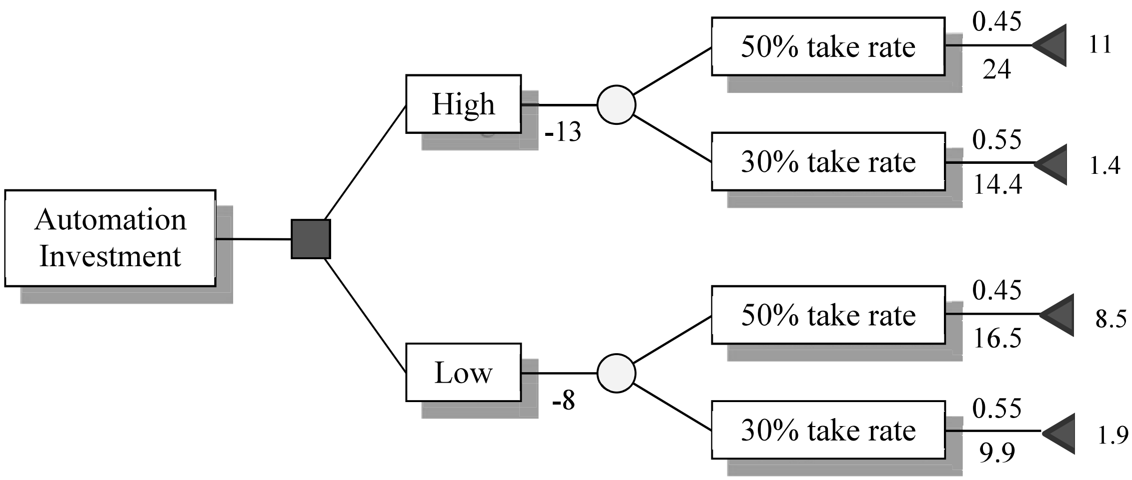



**Figure 3.2.5:** Boss Controls decision tree updated with values at end nodes

The next step is to determine the expected value of the net profit for each decision. We simply use the basic formula for expected value. The expected value of the net profit for a high investment is calculated as follows:

> If we make the high investment decision, there is a 0.45 probability that the net profit will be $11 million. There is a 0.55 probability that the net profit will be only $1.4 million. Thus, the expected value of the net profit for the high investment decision is

Expected Net Profit = E(N) = (0.45)($11 million) + (0.55)($1.4 million) = $5.72 million

10. What is the expected value of the net profit for a low investment?

11. Is it clear from your answers whether BC should make a low or high investment in automation? Why or why not?

12. Which of the two options, high or low investment in automation, has the larger expected value for net profit? How much larger than the other option is it?

13. Suppose the vendor for the high-speed automation has just announced a $500,000 price increase from $13 million to $13.5 million. Would this affect the preferred decision? What if the increase was $1,000,000?

### 3.2.3 Boss Controls Sensitivity Analysis

The management at Boss Controls now wonders how sensitive the solution to their automation problem is to their probability estimates of the take rate. Recall that they have estimated the probability of a 50% take rate to be 0.45 and the probability of a 30% take rate to be 1 – 0.45 = 0.55. But, they wonder, "What if our estimate of the probability of a 50% take rate is too high? Could that change which alternative has the larger expected value of profit? If so, at what point does the larger expected value change from high investment to low investment?" In other words, they want to learn how much the probability of a 50% take rate could change and still have the high investment in automation as the preferred alternative with the larger expected value of the profit.

14. Notice that the BC managers are not wondering what would happen if their estimate of the probability of a 50% take rate is too low. Why not?

The decision to make either a high or low investment in automation rests on the expected value of the profit for each of the alternatives. Recall that those expected values are given by the calculations shown below.

Expected profit for high investment = (0.45) ($11 million) + (1 – 0.45) ($1.4 million)
= $5.72 million

Expected profit for low investment = (0.45) ($8.5 million) + (1 – 0.45) ($1.9 million)
= $4.87 million

Now, for each alternative, we can define a function to calculate the expected value of the profit based on varying probability estimates.

Let: $x$ = the probability of a 50% take rate,
$y_1$ = the expected profit for high investment, and
$y_2$ = the expected profit for low investment.

Then, $1-x$ = the probability of a 30% take rate,
$y_1 = (x)(11) + (1-x)(1.4)$ and
$y_2 = (x)(8.5) + (1-x)(1.9)$.

Notice that in both cases we have dropped the "$" and the "million." We can do this, because we are using the same unit of measure, millions of dollars, in each case.

Figure 3.2.6 shows the set-up and calculator graphs of these two functions on the same coordinate axes. Use a graphing calculator to graph this system of equations on the viewing window shown.


**Figure 3.2.6:** Expected value functions, window, and graphs for sensitivity analysis

15. The viewing window for the graphs has been set up so that $x$ varies between 0 and 1, only. Why does that make sense for this problem?

16. What do the $y$-values of the functions represent? What are their units of measure?

Use the TRACE feature of your calculator to trace on the graph of $y_1$ until the value of $x$ is 0.45 when rounded to two decimal places.

17. When $x \approx 0.45$, is $y_1$ above or below $y_2$? What does that mean in the context of the problem?

18. Use the calculator to find the expected value of profit for high investment if the probability of a 50% take rate is 0.4? What is the expected value of profit for low investment for this probability of a 50% take rate?

19. What are the slopes of $y_1$ and $y_2$, the expected value of profit lines? What do these slopes mean in the context of the problem?

20. Notice that the graphs of the two functions intersect. What is the significance of the point of intersection?

21. Use the calculator to find the $x$-value of the point of intersection, rounded to two decimal places. What does this $x$-value tell you?

22. For what probabilities of a 50% take rate does the high investment in automation produce the higher expected value of profit? For what probabilities does the low investment produce the higher expected value?

23. Set up an equation to determine the $x$-value of the intersection point.

24. Use the equation to determine the $x$-value of the intersection point.

# Section 3.3 Green Tree Energy – Locates New Plant

Green Tree Power, Inc (GTP) is planning to expand their energy company by building a new power plant in a developing country. After much consideration, they have narrowed their possibilities to the countries of Cassedonia and Kisanthia. Each country can provide the required land and utilities for GTP to build and run their new power plant. In turn, the selected country will gain the benefits of the new energy technology that GTP can provide. Choosing the ideal location relies on several key pieces of information.

GTP estimates that the investment cost will be $50 million to build the new power plant in Cassedonia. However, there is significant uncertainty with regard to increased demand for power. As a result, the predicted total net revenue over the next five years is uncertain. Experts project that revenues could be as low as $80 million or as high as $110 million. The specific probabilistic forecast is that five-year total net revenues will be $80, $90, or $110 million with a 30%, 40%, and 30% chance, respectively. The political structure in Cassedonia is in transition. There are multiple political parties fighting for control of the country. These political parties have very different social and economic plans. Thus, the leadership of GTP believes there is a 20% chance that a Cassedonian government will take over the new power plant. If so, the government will simply repay the original investment cost of $50 million with no interest. In that case, GTP would have gained no net revenue from its investment.

If GTP builds the new power plant in Kisanthia, the investment cost is still $50 million. The population of Kisanthia is slightly lower than in Cassedonia and demand is still uncertain. GTP estimates the total net-revenue for a five-year period in Kisanthia, after the operation costs, will be $66, $80, or $90 million with a 30%, 40%, and 30% chance, respectively. The country of Kisanthia is a long established stable democracy that is committed to encouraging foreign investments. While the total forecasted revenue in Kisanthia is significantly lower than in Cassedonia, it has a major advantage. There is little chance that the Kisanthia government will take over the new power plant.

Where should GTP build their new power plant?

## 3.3.1 Develop the Decision Tree

Joe Riden, a senior risk analyst from GTP, has been charged with making this decision. He wants order to weigh the options carefully and objectively. Joe will create a decision tree to analyze the situation and determine the best choice for GTP.

In this situation, there is one decision to make with two possible options—GTP's new power plant can be built in Cassedonia or Kisanthia. Joe decides to develop the decision tree in stages. First he places the two alternative decisions at the first decision node. The uncertainty with regard to Kisanthia involves just the net revenue. He therefore, adds the random event, net revenue, to the Kisanthian branch of the tree. He includes all of the critical information at the appropriate places. He places a -50 on the decision branch to represent the investment cost. The net revenue random event has three branches, high, medium, and low. For each branch, Joe inputs the probability and the net revenue values. For the High revenue branch the probability is 0.3 and its projected revenue for Kisanthia is $90 million. He also inputs the corresponding numbers for Medium (0.4, $80 million) and Low (0.3, $66 million). This first stage of Joe's tree construction is presented in Figure 3.3.1.

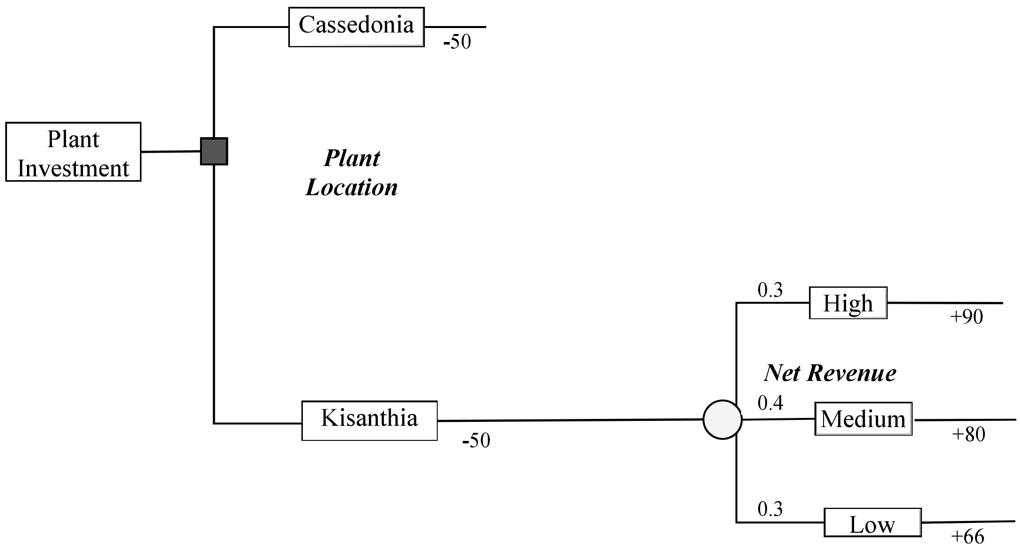



**Figure 3.3.1:** GTP – Kisanthian alternative

Next Joe adds the uncertainties related to building a plant in Cassedonia. Recall there is a 20% chance that the Cassedonian government will take over the power plant after it is completed. We therefore need a chance node off of Cassedonia to represent this possibility. The random event, a government takeover, has two branches: yes and no. If the government seizes control of the power plant, they will repay the $50 million dollar investment and GTP will no longer be involved with the plant. This return of $50 million is included along the Yes branch for a government takeover.

However, there is an 80% chance that the government will not take over the new power plant and GTP can move on to producing power, and thus revenue. Down this branch, there is another random event, uncertain net revenue. Joe attaches another random node with three branches for the net revenue. Again, each branch has a probability and dollar amount for net revenue: High (0.4, $110 million) Medium (0.4, $90 million) and Low (0.3, $80 million). Joe added the probability of a takeover and the net revenue uncertainty as presented in Figure 3.3.2.

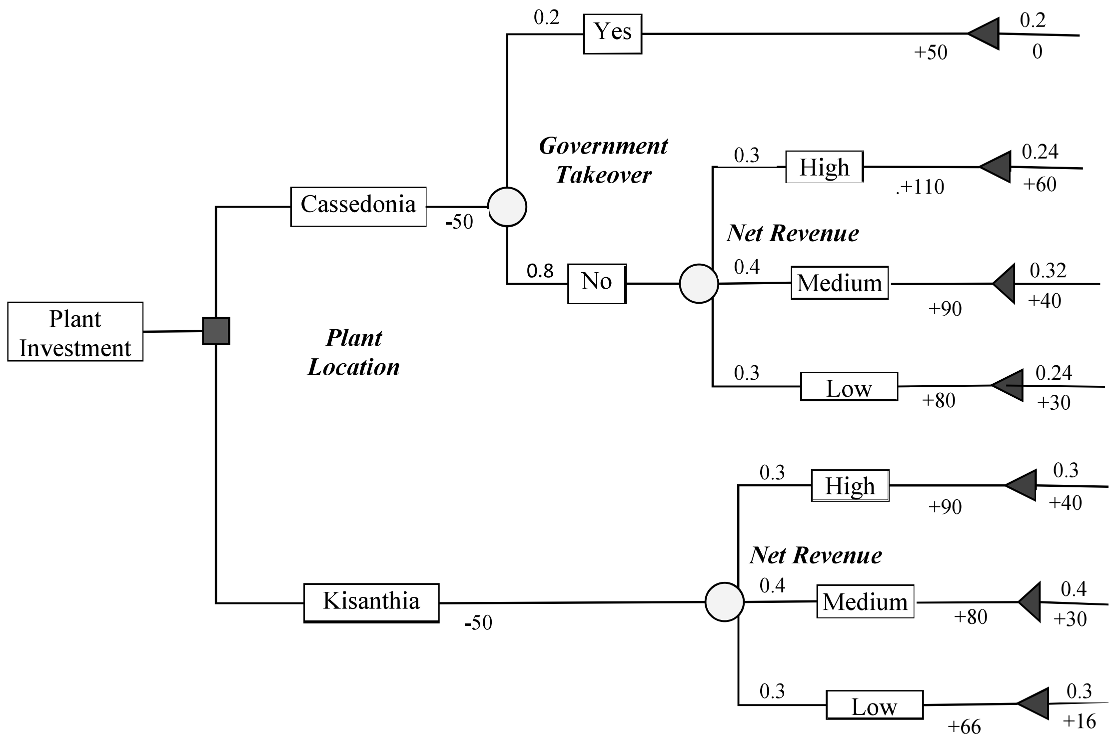



**Figure 3.3.2:** GTP – Cassedonian alternative added

In order to make the decision, we need to perform some calculations. At each end node, we want to note two important values: the probability of following that path from start to finish and the total profit for GTP if they follow that path. For example, assume GTP were to build in Kisanthia and net revenues turn out to be high. The net profit would be $40 million: $90 million in net revenue minus $50 million investment. The probability of that happening is just 0.3. He then adds the probabilities and net profits for the other two branches.

For Cassedonia, if the government takes over the plant, the net profit is zero. This has a probability of 0.2 of occurring. If the government does not take over the plant and the net revenue is high, the net profit is $60 million ($110-$50). The likelihood of this happening is the product of two probabilities. Mr. Riden multiplies the probability of no government takeover, 0.8, by the probability of high net revenue, 0.3. The product is 0.24. He repeats this process for the other two possible branches for net revenue for Cassedonia. Figure 3.3.3 contains all of the final dollar values and probabilities placed alongside the end nodes. The probability is placed above the net profit value.

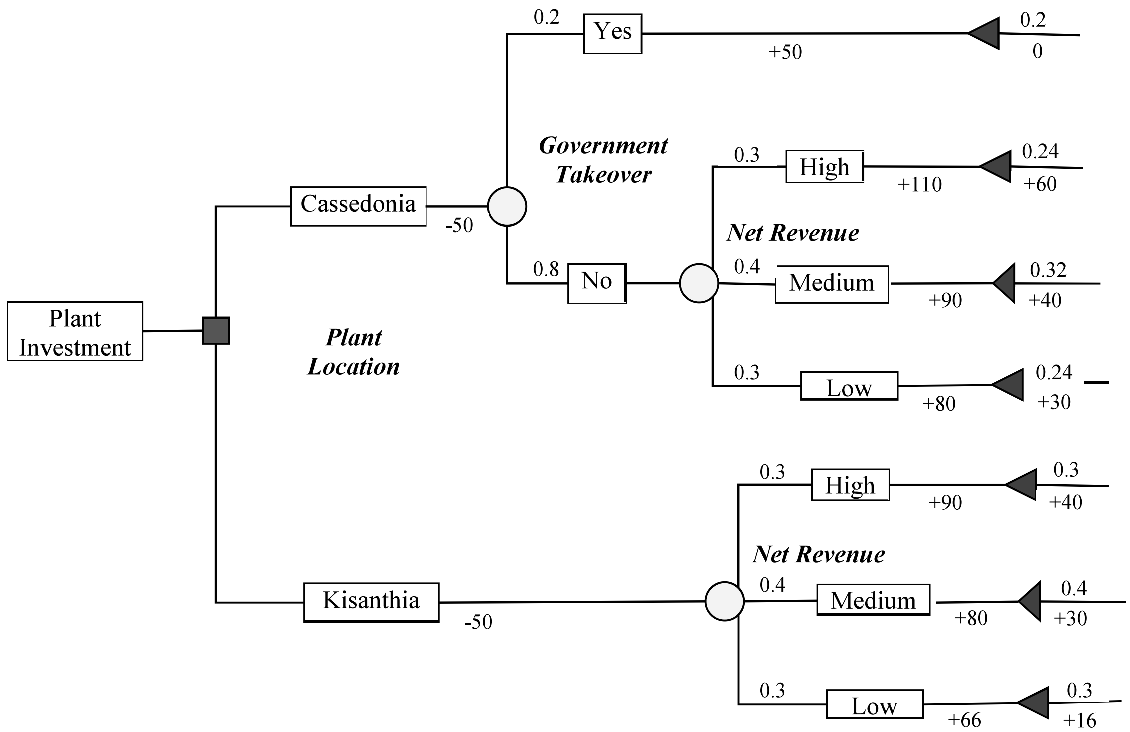



**Figure 3.3.3:** GTP complete decision tree

## 3.3.2 Compute the Expected Value

The final step in making the decision is to determine the expected value of the net profit for each alternative decision. We need the expected value of the net profit for building the power plant in Cassedonia and the expected value of the net profit for building in Kisanthia. The country with the larger expected value will be the best option for GTP.

The expected value for net profit of building the plant in Kisanthia is determined by taking the weighted sum of the different possible net profit values. There is a 30% chance of making $40 million, a 40% chance of making $30 million, and a 30% chance of making $16 million. So, the expected value of net profit is calculated as shown below.

$$E(N) = (0.3)(\$40 \text{ million}) + (0.4)(\$30 \text{ million}) + (0.3)(\$16 \text{ million}) = \$28.8 \text{ million}$$

Consider the option of building in Cassedonia. There is a 20% chance of making $0, a 24% chance of making $60 million, a 32% chance of making $40 million, and a 24% chance of making $30 million. Thus, the expected value of net profit for building the plant in Cassedonia is shown below.

$$E(N) = (0.2)(\$0) + (0.24)(\$60 \text{ million}) + (0.32)(\$40 \text{ million}) + (0.24)(\$30 \text{ million})$$
$$= \$34.4 \text{ million}$$

1. Based on the expected values, which country is preferred for the investment?
2. What is the minimum net profit for Cassedonia? How likely is that to occur?
3. What is the minimum net profit for Kisanthia? How likely is that to occur?
4. How might the issue of risk affect your preferred decision?

### 3.3.2 GTP Considers Insurance

Freud's of London understands the psychology of risk. It offers specialty risk insurance for large projects. Freud's is prepared to offer GTP insurance against a possible government takeover in Cassedonia. They are prepared to charge GTP a premium of $3.5 million to insure against a government takeover. If the government takes over GTP plant, Freud's will pay GTP $10 million dollars. We will analyze this insurance policy both from GTP's perspective and that of Freud's.

The GTP part of the tree is revised and presented in Figure 3.3.4. A negative $3.5 million dollars for the insurance premium is added to the cost of the Cassedonia branch. That total cost is now $53.5 million. As a result each end value for NO government takeover decreases by $3.5 million. However, the end node for the YES government takeover now has a net profit of $6.5 million dollars. They receive $50 million back from government plus a $10 million payment from the insurance company for total revenue of $60 million. The cost of investment plus the insurance premium is $53.5 million. The net profit would be $6.5 million.

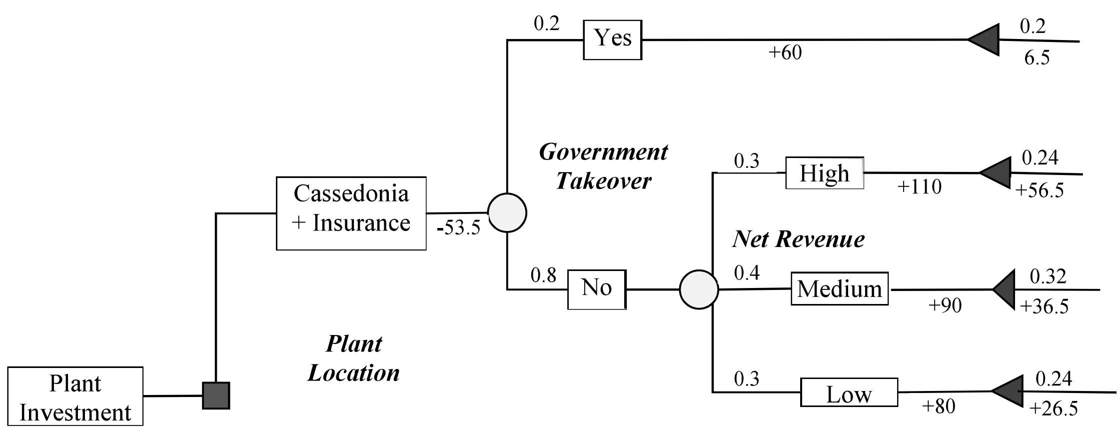



**Figure 3.3.4:** GTP Cassedonia with insurance

The new expected value of net profit for the Cassedonian option can be calculated. The equation shown below utilizes the abbreviation $6.5M to represent 6.5 million dollars.

$$E(N) = (0.2)(\$6.5M) + (0.24)(\$56.5M) + (0.32)(\$36.5M) + (0.24)(\$26.5M) = \$32.9M$$

The expected net profit has declined by $1.5 million. However, GTP is assured now of making at least $6.5 million from its $50 million investment in Cassedonia.

5. Would you recommend buying the insurance?

The above analysis focused on GPT's perspective. Let's look at the decision from the insurer's perspective, Freud's of London. If they offer insurance, they face only one uncertainty, a government takeover. They are not impacted by the uncertain event, net revenue. They are only insuring against a government takeover. Their decision tree is presented in Figure 3.3.5. If they sell no insurance, they gamble no money and they make no profit. If GPT purchases insurance and there is a government takeover, Freud's must pay GPT $10 million. Their net loss would be $6.5. With no takeover, their profit is the $3.5 million insurance premium they charged.

6. How much risk does Freud's face?

7. What is the expected value of the profit Freud's will earn?

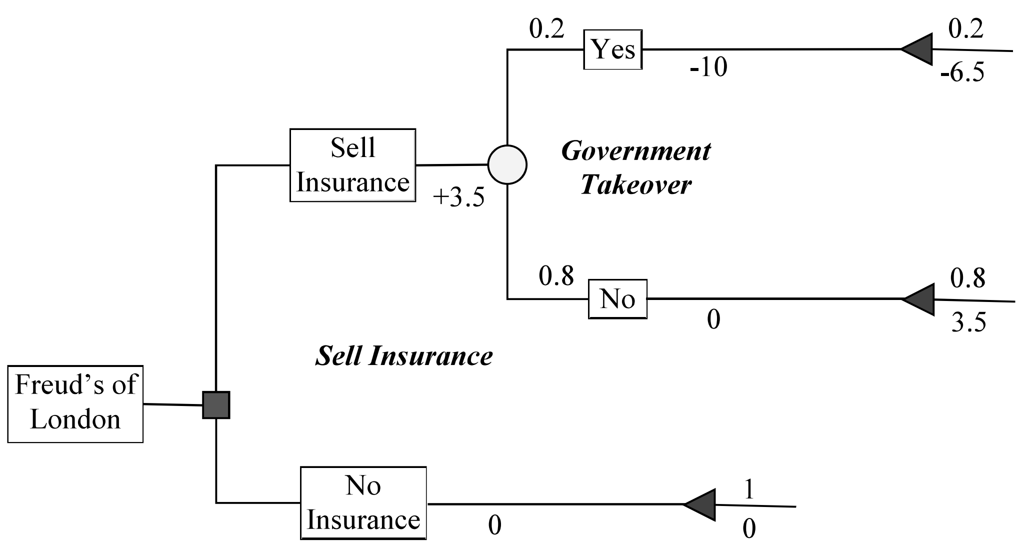


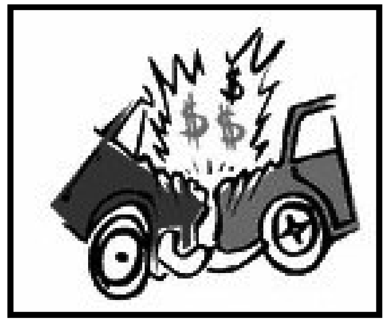

**Figure 3.3.5:** Freud's of London insurance decision

The early decision analysts recognized that expected value alone did not represent the way many individuals and companies deal with risk. Many of us are "risk averse." This means we would be willing to accept a reduced expected value in exchange for more certainty. This is the reason people buy insurance. They are willing to pay money to avoid risk. The cost of the insurance is always more than the expected value of the loss.

We pay hundreds of dollars to insure against a relatively unlikely catastrophic risk to our homes. We buy family medical insurance for thousands of dollars to cover the cost of possible major surgery and a long hospital stay that could cost more than a hundred thousand dollars. As individuals living one life and facing one situation, we cannot rely on long range expected value. However, insurance companies can tolerate these types of risks and live by the expected value. They can pool the risk across thousands of customers each year. Their financial performance will approximate the expected value.

Decision analysts developed a concept called utility theory to quantify this concept of risk aversion. The mathematics of utility theory is beyond the scope of the course. Instead, we will present tradeoffs between reduced expected value and more certainty and let you judge your preference. In the next example of automobile collision insurance, we will challenge you to revisit your own attitude towards risk when making smaller insurance decisions

## Section 3.4 Purchase Collision Insurance





Jee Min is a high school junior at Cassidy High School in Thomasville, Michigan. He has been an excellent driver for one year. With the help of his parents, he has just purchased a 2010 Chevrolet HHT. Jee drives his car to school and to his part-time job on the weekends and after school. He is considering purchasing collision insurance for this car. It has a *Kelly Blue Book* value of $6,600. Jee gathered quotes from insurance companies through the internet. He learned that the lowest six-month premium for collision insurance with a $500 deductible is $1,700.

He is not sure that he can afford that much, so he decides to investigate the cost of collision insurance with a $1,000 deductible. The six-month premium for collision insurance with a $1,000 deductible is $1,500. The deductible represents the maximum amount of loss the owner incurs in the case of an accident. With a $500 deductible, the owner must absorb the cost of the first $500 of damages. For example, if the damages were $250, he bears the whole cost. If the damages were $6,600, he absorbs the $500 and the insurance company pays him $6,100. Jee Min is also considering the possibility of not carrying any collision insurance. In order to make the best decision, he must consider all of the consequences of each possibility.

To examine his options, Jee created a decision tree showing the three collision insurance possibilities. Each branch of the tree is labeled. Let's examine his decision tree. The tree begins with a decision node. Jee must decide whether to purchase collision insurance with a $500 deductible, a $1,000 deductible, or no collision insurance at all. The next node on each of the three branches is a probability event: a collision in the next six months. He chose a six-month time period for this event, so that it matches the six-month period covered by the premium quotes that he obtained. Next, Jee Min needed an estimate of the probability that he would be involved in a collision in the next six months.

After some Internet research, Jee Min learned that according to the National Highway Safety Administration, the probability that a male teenage driver in the U.S. will have an accident in any six-month period is 30%. He decided to use this probability estimate. Then he attached branches to the probability nodes to account for the possibilities that he will have an accident or not during a given six-month period.

1. How do you think the National Highway Safety Administration determined the probability a male teenage driver will have an accident in a six-month period?

2. Why was 70% assigned to the branches representing the event that Jee Min will not have an accident in the next 6 months?

Let's explore the structure of the tree (See Figure 3.4.1).

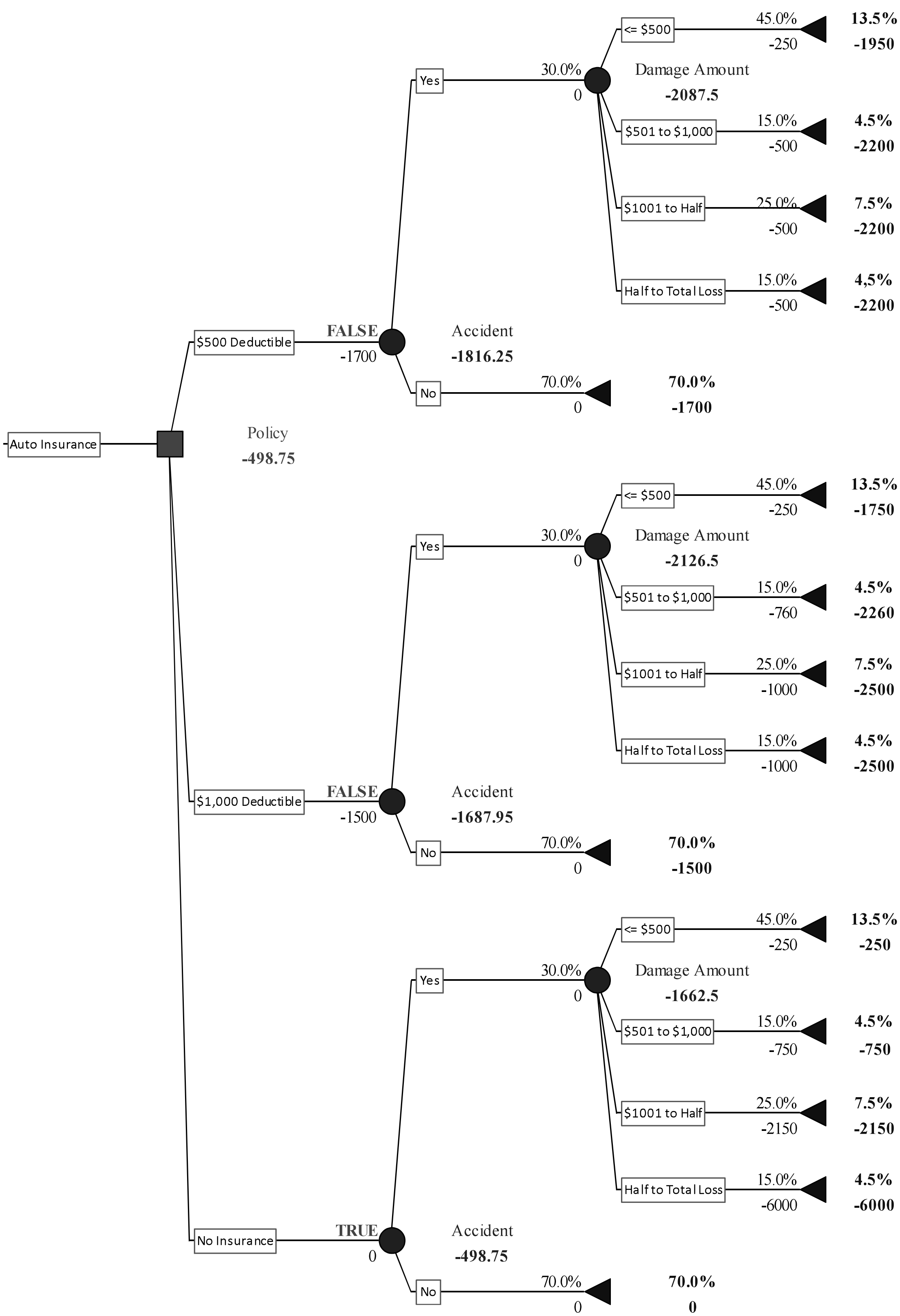



**Figure 3.4.1:** Collision insurance decision tree

3. There are three branches that leave the main rectangular decision node. What do these three branches represent?

4. Each of these three branches leads into a random circle node. What does that random node represent? Why are there two branches coming out of this node?

If Jee Min does have an accident, various damage amounts are possible. He realized that even if he rounded damage amounts to the nearest dollar, there would still be $6,600 possible damage amounts. Therefore, he decided to list these amounts in ranges. He also investigated the probabilities that the damage amount of an accident falls within that range. Jee Min also collected this probability data online from the National Highway Safety Administration. He organized this information in Table 3.4.1.

| **Damage Amount Ranges** | **Probability** |
|---|---|
| Less than or equal to $500 | 45% |
| Greater than $500 but less than or equal to $1,000 | 15% |
| Greater than $1,000 but less than or equal to one-half the value of the car | 25% |
| Greater than one-half the value of the car but less than or equal to $6,600, the total value of the car | 15% |

**Table 3.4.1:** Damage estimate probabilities

Based on these data, Jee Min added this information to the decision tree. He added another probability node to each branch of the tree that represents having an accident in the next six months. Then he added a branch for each of the possibilities in Table 3.4.1 to each new node. Finally, he labeled each new branch with the range of an amount of damage. He also assigned the corresponding probability of occurrence. Note the number of sequences of branches in Jee Min's tree. The probability of that sequence of random events is listed at the end of each sequence of branches.

5. Each branch labeled YES contains another random node. What does this represent? Why are there four branches leaving this second node?

Let's explore the probabilities on the tree. There is a probability assigned to the end node of each sequence of branches. For example, the top end node has a 13.5% assigned to it.

6. How was this end probability determined?

7. Which end nodes have the highest probabilities and why?

8. The probabilities for the top five branches in the picture sum to 1. Why is this the case?

Let's discuss the dollar amounts of the tree and the end node values. Jee Min realized that for each complete decision branch of his tree, he must assign a cost to each branch. However, in order to do these calculations, he realizes that he needs *individual* damage amounts and not *ranges* of damage amounts. Jee Min decides to use a single number within each range to represent the damage amount for that branch. He listed those in the following table.

| Damage Amount Ranges | Representative Amount |
|---|---|
| Less than or equal to $500 | $250 |
| Greater than $500 but less than or equal to $1,000 | $750 |
| Greater than $1,000 but less than or equal to one-half the value of the car ($3,300) | $2,150 |
| Greater than one-half the value of the car but less than or equal to the total value of the car | $6,600 |

**Table 3.4.2:** Damage estimate – representative amounts

For the three lowest damage ranges, Jee Min decided to use the midpoint of the range: $250 for accidents having damage less than or equal to $500, $750 for accidents having damage greater than $500 and less than or equal to $1,000, and $2,150 for accidents having damage greater than $1,000 and less than or equal to $3,300, one-half the $6,600 value of his car. For the last range, damage greater than one-half value and less than or equal to the full value of his car, he learned that insurance companies almost always "total" the car when the damage falls within this range. Therefore, he decided to use $6,600 for accidents in that range.

All four of these dollar amounts are entered onto the appropriate branches of the tree for the no collision insurance decision. Now let's look at the section of the tree with the decision, $1,000 deductible, the random event "yes", and the random event "damage". The dollar amounts for these four branches do not match the values in the table.

9. Which two branches match the table and which two do not? Why?

10. For the section of the tree with a $500 deductible, there are three branches with a $500 value. Why is this the case?

Consider only the portion of the entire decision tree that represents the decision to purchase $500 deductible collision insurance. Recall that the premium for this deductible amount is $1,700. For the top-most branch, the cost is $250. Thus, the end node value is the sum of these two costs, $1,950. The second branch, has a $500 cost. The end node value is therefore $2,200.

11. Why are the end node values for the third and fourth branch the same $2,200 as for the second branch?

In a similar way, Jee Min adds the cost of the premium with his liability for damages to determine the end node value for each sequence of branches.

12. What is the expected cost to Jee Min of the decision to purchase $500 deductible collision insurance?

13. Examine each of the remaining decision options and the total cost. In a similar way, determine the expected cost to Jee Min of making that decision. Enter each of the expected costs in the table below.

| Decision | Expected Cost |
|---|---|
| $500 Deductible | |
| $1,000 Deductible | |
| No Insurance | |

**Table 3.4.3:** Expected costs

14. Based on this analysis, which option has the smallest expected cost?

15. Should Jee Min base his decision only on this analysis? Explain why or why not.

The basis for the insurance industry recognizes the concern people have with incurring huge costs associated with relatively infrequent events. Every reduction in premium fees can be directly subtracted from the total expected value to determine the impact.

16. Given your own attitude towards risk, would you be willing to pay more than the expected value calculated for the no insurance options to reduce your risk? If so, how much would you be willing to pay to have a policy with a $1,000 deductible?

## 3.4.1 A Revised Estimate of the Probability of Having an Accident

Over the past six years, Michigan and other states in the U.S. have instituted graduated driver training programs for new teenage drivers. In fact, Jee Min participated in such a program. The establishment of these programs has resulted in safer teenage drivers.

What if the National Highway Safety Administration has now determined the probability that a teenager who graduated a driver trainer program will have an accident in any six-month period is 22%.

17. Will this new estimate of the probability of having an accident affect the expected cost for each decision?

18. Using this new estimate, what is the probability of *not* having an accident in the six-month period?

The insurance companies are slowly responding to this reduced rate of accidents. They are considering significant reductions in the premiums charged.

19. If the premiums were reduced by 25%, what would be the expected values for each of the insurance policies?

20. Recalculate the expected cost for no insurance option. Should Jee Min change his decision? Explain.

# Chapter 3: (Decision Trees) Homework

1. The probability of selling a dress in a store is 25% each week.
   a. Construct a probability tree to determine all of the possible outcomes over a three-week period.
   b. What is the probability that the dress will not be sold at the end of the third week?

2. A contestant on a TV show must pass four stages to win a big prize. The probabilities of winning in stage 1, 2, 3, and 4 are 0.8, 0.6, 0.4, and 0.3 respectively. The contestant wants to know the probability of winning the big prize.
   a. Construct a probability tree to determine the possible outcomes of the game.
   b. What is the probability that he wins the big prize?
   c. What is the probability that he makes it to stage 4 but does not win the big prize?

3. A TV cable company has a technical support department to solve customers' problems by phone. In this department the staff members are categorized into four levels based on their ability to solve customer problems. The company first assigns a problem to Level 1; if they cannot solve it, someone at Level 2 is assigned. This process is repeated until it finally reaches the most experienced staff for one last attempt at solving the problem. The probability that a staff person is able to solve the problem at each Level is 0.50, 0.75, 0.85, and 0.95 consecutively.
   a. List all of the possible outcomes.
   b. Construct a probability tree to determine the probability of each outcome.
   c. What is the probability that a customer's problem is unsolved?
   d. What is the probability that the problem is solved by someone at Level 3?
   e. Why is this probability less than the probability that a Level 1 individual solves the problem?

4. A manager at Wayne State football games must decide 10 days in advance which product to order for the stadium vendors to sell. Each product will have the university logo. The three options are sunglasses, umbrellas, and ponchos. He will stock only one of the items. Sales and the resultant profit will depend upon the weather on the day of the game. The long-range weather forecast is 35% chance of rain, 25% chance of overcast skies, and 40% chance of sunshine. Table 1 contains the manager's estimates of the profits that will result from each decision and each weather condition.

| **Decision** | **Profit ($)** | | |
|---|---|---|---|
| | **Weather Condition** | | |
| | Rain (0.35) | Overcast (0.25) | Sunshine (0.40) |
| Sunglass | -600 | -300 | 1,600 |
| Umbrella | 2,100 | 0 | -800 |
| Poncho | 1,800 | 500 | -600 |

**Table 1:** Wayne State sales of items with logo

a. What is the best decision for each weather condition?
b. Draw the associated decision tree needed to make the best decision.
c. What decision should be made if he desires to maximize the expected value?

5. The owner of a restaurant is considering two ways to expand operations: open a drive-thru window or serve breakfast. There are increased annual costs which each option and a one-time cost associated with the drive-thru. Labor and marketing costs are annual costs that the restaurant has to pay each year. They include hiring new staff and placing more ads in local media. Redesigning the restaurant is a one-time cost that is paid at the beginning and does not repeat each year. The details are provided in the following table.

| Decision | Costs ($) | | |
|---|---|---|---|
| | Annual | | One Time |
| | Labor | Marketing | Restaurant Redesign |
| Drive-thru window | 28,000 | 10,000 | 20,000 |
| Breakfast | 38,500 | 5,000 | - |

**Table 2:** Restaurant expands operations - costs

The forecasted increase in income resulting from these proposed expansions depends on whether a competitor opens a restaurant down the street or not. Based on the restaurants evaluation, the manager believes that the competitor won't open a new restaurant with 60 percent probability. Based on the competitor action, the restaurant's profit will be different for each decision. The following table provides an estimation of the increase of income based on the competitor's action.

| Decision | Revenue ($) | |
|---|---|---|
| | Competitor | |
| | Open (0.4) | Not Open (0.6) |
| Drive-thru window | 110,000 | 130,000 |
| Breakfast | 80,000 | 120,000 |

**Table 3:** Restaurant expands operations - revenue

The owner of the restaurant is focused just on next year. He therefore decided to consider the one-time cost for the redesign the same as all of the labor and marketing costs that are ongoing.

a. Calculate the profit of each decision when considering the competitor's action.
b. What is the best alternative if no competitor opens nearby? What is the best alternative if a competitor opens nearby?
c. Draw the associated decision tree.
d. What decision should the company follow?
e. Let $p$ represent this probability that the competitor will open a restaurant down the street. Write an equation to calculate the expected value for each decision as a function of $p$.
f. For what value of $p$ are these two expected values equal?

g. Graph the equations of the expected values to determine their intersection point. What does this intersection point represent?
h. Recall that the owner treated the design change and marketing cost the same as operating costs. Would the decision change if he considered only 50% of these costs this year (design and marketing)?

6. A company is about to launch its new fast food for sale in supermarkets throughout Arkansas. The research department is convinced that a special type of chicken wings will be a great success. The marketing department wants to launch an intensive advertising campaign. The advertising campaign will cost $1,000,000 and if successful will produce $4,800,000 profit. If the campaign is unsuccessful (25% chance), the profit is estimated at only $1,800,000. If no advertising is used, the revenue is estimated at $3,500,000 with probability 0.6 if customers are receptive and $1,500,000 with probability 0.4 if they are not.
   a. Draw the associated decision tree.
   b. What course of action should the company follow in launching the new product if they want to maximize the expected value?
   c. Write an equation to calculate the expected value for each decision as a function of the probability that the major advertising campaign will be effective? (Hint: one of the alternatives is unaffected by this probability.)
   d. Graph the equations of the expected values to determine their intersection point. What does this intersection point represent?

7. A discount clothing store uses an interesting strategy to attract customers to return each week to shop. They tell the customers that every 7 days they reduce the most recent price of an unsold dress by an amount equal to 25% of the original price. On each dress there is a label of its original price and the date it was hung on the rack. Thus customers know that a $40 dress placed out on Nov. 7$^{th}$ will be priced only $30 on Nov. 14$^{th}$ if it is not sold before then. It will be reduced by another $10 on Nov. 21$^{st}$ if it is still unsold. After three weeks, any unsold dress is sent to a local charity. Each week, there is a .60 probability that the dress will be sold.
   a. Nancy Drew saw a dress she really liked and knows she can get the almost identical dress for $50 online. The current store price is $40. Construct a decision tree to determine whether or not she should buy the dress now or gamble and wait a week. She would buy it next week if it remains unsold. (If when she comes back next week, Nancy finds the dress has been sold, she will buy it online.)
   b. Just before finalizing her decision, she found another place online that sells the same dress for $45. Why might a lower price online affect her purchase decision in this store? Should she buy the dress now or gamble and buy it in the second week if available?

c. She just saw a more expensive dress for sale at \$80. These more expensive dresses have only a 40% chance of being sold each week. Again, they tell the customers that every 7 days they reduce the most recent price of an unsold dress by an amount equal to 25% of the original price. Assume she would be able to buy a similar dress for \$90 online. Construct a full tree for 3 weeks and specify the optimal decision.

8. A contestant on a TV show has to decide whether to stop or try to answer another question. The contestant is first asked a question about US Geography. If the contestant answers correctly, she earns \$700. Historically, three out of four contestants answer the first question correctly. If answered incorrectly, the game is over. If answered correctly, the contestant can leave with \$700 or go on and answer a question about US presidents. If answered correctly, the contestant wins an additional \$1000. If the answer is incorrect, the contestant loses all previous earnings and is sent home. Historically, two out of three contestants answer this question correctly. The third question is about rock 'n' roll music. This question is worth \$1500, and the same rules apply. The chance of answering this question correctly is 50-50.
   a. Draw a decision tree that can be used to determine how to maximize a contestant's expected earnings. What is the best decision and what are the expected earnings in this case?
   b. Some contestants may feel more or less knowledgeable about the third question category. Let $p$ represent the probability that a contestant will answer the third question correctly. Write an equation to calculate the expected value for attempting to answer the third question in terms of $p$.
   c. Based on the previous question, what is the cutoff value of $p$ such that a contestant should attempt the third question?
   d. The TV show is considering changing the reward for answering the 3rd question correctly. Let $m$ represent the amount of money a contestant will earn for answering the third question correctly. Write an equation to calculate the expected value for the last decision as a function of $m$.
   e. Graph the equations of the expected values to determine the intersection point for the last stage. What does this intersection point represent?

9. SSS Company, a software company, is considering submission of a bid for a state government contract to install their software on 30,000 computers. The government would use their software to oversee the management of tens of thousands of large and small contracts the government signs every year. There is only one other potential bidder for this contract, Complexo Computers, Inc. Complexo has a long record and reputation with this kind of contract. As a result of its lesser experience, to win the bid SSS's bid must be at least \$5 less per computer installation than Complexo's. Complexo Computers is certain to bid and is generally more expensive than SSS. SSS management believes that it is equally likely that Complexo will bid \$100, \$90, or \$80 per computer installation.
   a. What are the possible bids that SSS should consider?

SSS's bidding decision is complicated by the fact that it is currently working on a new process to install software remotely through the internet. If this process works as hoped, then it may substantially lower the cost of installations. However, there is some chance that the new process will actually be more expensive than the current installation process. Unfortunately, SSS will not be able to determine the cost of the new process without actually using it to install the software. The higher SSS bids the more money it makes if it wins the contract. However, the higher the bid, the less likely it is to win the contract. If SSS decides to bid, it will cost $20,000 to prepare all of the relevant documents required to submit the bid. SSS will incur this expense regardless of whether it wins or loses the bidding competition. With the proposed new installation process, there is a 0.25 probability that the cost will be $50 per computer and a 0.50 probability that the cost will be $75 per computer. Unfortunately, there is also a 0.25 probability that the cost will be $85 per computer.

b. Construct a decision tree to model this situation.
c. Based on your decision tree, do you recommend that SSS Company submit a bid. If so, what should they bid per installation?
d. Under the optimal policy, what is the probability they will win the contract?
e. What is the overall expected value if they bid on the contract?
f. If they win the contract, what is their expected value of profit? (Hint: This is a conditional decision analysis.)

10. A group of high school students has decided to start a summer business. They are thinking about designing and coloring T-shirts and selling them to clothing stores in their community. For mass production of colored T-shirts, they need special equipment which they can buy or rent. After negotiating with a company about equipment, they figure out that they have three options to start their business:

- They can buy all of the equipment and do the design and printing themselves. In this case they have to pay for the equipment but they can recover part of the money at the end of summer by reselling the equipment. The cost of buying the equipment is $8,100. They can resell it at 50% of the original price. The cost of printing will be $1 per T-shirt.

- The second option is renting the equipment and returning it at the end of summer. The renting cost is $1,500 for the whole summer with a variable cost of $1.50 per print.

- The third option is outsourcing the printing. In this case they do the designs themselves but send them to a company for printing. The company charges them $2 per T-shirt.

In each option the unprinted T shirt costs them one dollar.

The fact that the market demand for colored T-shirts is not certain makes the decision making difficult. After doing some market evaluation, they summarized their expectation in following table.

| Demand (Number of T-shirts) | Probability |
|---|---|
| 2,000 | 15% |
| 5,000 | 50% |
| 8,000 | 35% |

**Table 4:** Probabilistic demand for T-shirts

a. If they can sell each T-shirt for $5, construct a decision tree to help make the decision.
b. What is the best option if the demand is 2,000 T-shirts?
c. What is the best option if the demand is 5,000 T-shirts?
d. What is the best option if the demand is 8,000 T-shirts?
e. Which option is the best for them? What is the expected profit if that decision is made?

11. A software company released a beta version of a software package. It expects a large number of requests from the users for fixing potential bugs in the software. These include crashing, lock up, and incompatibility errors. The company has established a help desk to handle telephone requests. The company trained two groups of software specialists to support the software. Group 1 has just been hired and trained; meanwhile specialists in Group 2 are senior technicians very capable of solving the problems. The senior specialists solve the problems with 100 percent certainty, but their salaries are much higher than other specialists.

    The payment system of the company for the specialists is problem based. The company pays them based on the number of the problems that they attempt to solve. Group 1 salaries are $20 per problem and Group 2 salaries are $35 per problem. The software company always has a dilemma as to which specialist to assign in order to minimize the cost of the support. For example, assume they assign a problem to a Group 1 specialist. If he is not able to solve the problem, they reassign it to a senior specialist. In this case both specialists are paid. This costs the company $55 per problem. To address this issue, they developed an automatic system to predict the chance of solving a problem by Group 1 based on previous cases.

    a. A crashing problem was just received, and the prediction software forecasts a 70 percent chance of success for a Group 1 specialist. Draw a decision tree for this problem.
    b. What kind of specialist should be assigned to the problem first?
    c. Another problem, compatibility error, was received and the prediction software forecasts a 50 percent chance of success for a Group 1 specialist. Draw a decision tree for this decision.
    d. Based on the decision tree, what kind of specialist should be assigned to the problem?
    e. Let $p$ represent the probability that the Group 1 specialist will be able to solve the problem. Write an equation to calculate the expected value of the cost for each decision as a function of $p$.
    f. Graph the equations of the expected values to determine their intersection point.

g. They want to know what should be the cutoff value of the probability to assign directly to an expert instead of assigning the task to a Group 1 specialist. Use both a graphical representation of part e) and an algebraic representation of part f) to find that probability.
h. In the previous question, what was the role of the two salaries in determining the breakeven value of $p$? Assume that the salary of a Group 1 specialist is $x$ and the salary of Group 2 specialist is $y$. Write an equation to calculate the expected value of the cost for each decision as a function of $p$, $x$ and $y$.
i. Find the value of $p$ as a function of $x$ and $y$ that leads to the same expected value of the cost regardless of whether the problem is first assigned to a Group 1 or Group 2 specialist.

12. (Continuing the previous problem) After finishing the first phase, management figured out that they need another group of specialists that are more knowledgeable than Group 1 but not necessarily experts. They are to be paid at $28 because of their higher success rates than Group 1. This group is called Group 1.5.
    a. A problem was just received and the prediction software forecasts 70 percent chance of success for a Group 1 specialist. There is an 85 percent chance of success for a Group 1.5 specialist. Group 2 experts can solve the problem for sure. Draw a decision tree for this problem. (Assume that if a Group 1.5 specialist fails to fix the problem, the company will then assign it to Group 2.)
    b. Based on the decision tree, what kind of specialist should be assigned to the problem first?
    c. What is the optimal decision if the probability of success for Group 1 would be 60% and still 85% for Group 1.5?

13. An automotive part has to go through two different processes using metal lathes to be shaped properly. Each process has a cost associated with the type of lathe. Each step in processing has a risk of ruining the part and turning it into scrap. For example, when Lathe 1 processes a part, the cost is $100 and the risk of being scrapped is 10%. Each part that successfully processed by both lathes is sold for $450. The net profit is equal to the number of parts sold minus the cost of processing all parts. The cost of processing includes both finished and scrapped parts. The following table shows the cost and risk of each lathe.

| | Cost | Risk of scrap |
|---|---|---|
| Lathe 1 | $100 | 10% |
| Lathe 2 | $150 | 20% |

**Table 5:** Two lathe costs and scrap

There is no recycling value if the part is ruined; scrapped parts are worthless.
    a. What is the probability that a part will end up being scrapped? Does it make a difference as to the order of the processes?
    b. The processes can be done in either order. Draw a decision tree to determine the optimal sequence of processes which maximizes the expected value of the net profit per part.
    c. Which process should be done first?

d. If the cost of processing by Lathe 2 changed, for what cost would the optimal strategy change? What is this percentage change?

14. Continuing the previous problem, suppose a different part must undergo three processes to be done by three different lathes. Parts that are made with these 3 processes are sold for $800 each. Each lathe has an associated cost and risk of ruin as follows:

| | Cost | Risk |
|---|---|---|
| Lathe 1 | $100 | 10% |
| Lathe2 | $150 | 20% |
| Lathe3 | $200 | 25% |

**Table 6:** Three lathe costs and scrap

a. How many different sequences need to be considered?
b. What is the probability that a part is ruined?
c. Draw a decision tree to determine the optimal sequence of processes which maximizes the expected value of the net profit per part.
d. In which order should the processes be done?
e. Explain how you can use a pair-wise comparison among processes to find the optimal sequence?
f. If there were four processes, how many pair-wise comparisons would need to be made to find the optimal sequence?

# Chapter 3 Summary

## What have we learned?

We have learned that decision-making often involves uncertainty. It may be easy to choose between two events such as taking a job that pays $8 per hour or a job that pays $10 per hour. However, it becomes more difficult to choose if the benefits are less certain. What if the second job involved a lower base pay but you could earn tips. Imagine your base pay was $6 per hour but tips ranged from $0 to $4 per hour. Your decision would need to be based on how likely those amounts were. In order to choose between multiple options, we need to be able to identify not only the benefit of different possibilities but also the likelihood of the different possibilities occurring. A decision tree can help combine these together to find the expected values of different options. The option we should choose is the one that has the highest expected value. It is interesting to note that often the expected value is not actually a possible value.

Using probability decision trees is a multiple step process:

1. Create a decision tree including decision nodes, chance nodes and end nodes.
2. Identify the potential cost or benefit for each branch off of a node.
3. Calculate the cost or benefit for each complete path.
4. Identify the probability for each branch off of a chance node.
5. Find expected value for each chance node.
6. Compare the expected values for each branch of the decision node.
7. Select the decision node branch with the best expected value.

## Terms

| | |
|---|---|
| ***Arcs or branches*** | The connections between nodes on a decision tree showing alternative decisions or outcomes of random events. |
| ***Chance event*** | An event whose likelihood must be predicted using probability and is outside anyone's control. |
| ***Compound event*** | An event that is the combination of two or more simpler events. |
| ***Decision tree*** | A diagram similar to a probability tree used to model decision alternatives in which the profits or costs associated with the decision alternatives are affected by one or more random events. |
| ***Expected value*** | The weighted average found by multiplying the possible results of a random variable by the probability of the random variable having that value. |
| ***Fundamental principle of counting*** | If there are $m$ possible outcomes for one event and $n$ possible outcomes for a second event then there are $m \times n$ possible outcomes when both events occur. |
| ***Independent*** | Events $A$ and $B$ are independent if the outcome of event $A$ does not affect the outcome of event $B$, and the outcome of event $B$ does not affect the outcome of event $A$. |
| ***Multiplication rule*** | If $A$ and $B$ are independent events then the probability of $A$ and $B$ both happening equals the probability of $A$ times the probability of $B$. |
| ***Node*** | The boxes (representing decisions), circles (representing random chance), and triangles (representing the end of a path) on the decision tree. |
| ***Probability tree*** | A diagram used to show all the possible outcomes for a combination of two or more independent events. |
| ***Random variable*** | A variable whose numeric value depends on random events. |
| ***Risk aversion*** | The desire to avoid risks that can affect either costs or profits. |
| ***Take rate*** | The proportion of customers that select a particular option. |

# Chapter 3 (Decision Tree) Objectives

## You should be able to:

- Create a probability tree to calculate the probability of different sequences of events.
- Create a decision tree including decision nodes, chance nodes and end nodes.
- Identify the potential cost or benefit for each branch off of a decision node.
- Identify the probability for each branch off of a chance node.
- Use the fact that probabilities must add up to one.
- Find the value for each sequence of branches of the decision tree.
- Find the expected value for each decision option.

## Chapter 3 Study Guide

1. What is a decision tree and what is it used for?

2. What are nodes on a decision tree?  What shape is used for each type of node?

3. How do you find the end node value for a path in a decision tree?  Where do you put that value on the decision tree?

4. Once you find the value for each branch of a decision tree, how do you find the expected value for each alternative of the decision nodes?

5. What must be true about the sum of the probabilities of the branches after a chance node?

6. What does risk aversion mean and how does it affect the results of decision tree analysis?

7. Give an example of risk aversion.